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The 0-rook Monoid and its Representation Theory

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Abstract. We show that a proper degeneracy at $q = 0$ of the q-deformed rook monoid of Solomon is the algebra of a monoid R_n^0 namely the 0-rook monoid, in the same vein as Norton's 0-Hecke algebra being the algebra of a monoid $H_n^0 := H_n^0(A)$ (in Cartan type A). As expected, R_n^0 is closely related to the latter: it contains the $H_n^0(A)$ monoid and is a quotient of $H_n^0(B)$. It shares many properties with H_n^0, in particular it is $\mathcal{J}$-trivial. It allows us to describe its representation theory including the description of the simple and projective modules. We further show that R_n^0 is projective on H_n^0 and make explicit the restriction and induction along the inclusion map. A more surprising fact is that there are several non classical tower structures on the family of $(R_n^0)_{n\in\mathbb{N}}$ and we discuss some work in progress on their representation theory.

Keywords: combinatorics, representation theory of monoids, symmetric group, Iwahori-Hecke algebras, rook monoid

Introduction

The Iwahori-Hecke algebra were defined by Iwahori in [7] in the following way: let q be a prime power and let $M = \mathbf{M}_n(\mathbb{F}_q)$ be the monoid of all $n \times n$ matrices over $\mathbb{F}_q$. Let $G = \mathbf{GL}_n(\mathbb{F}_q) \subset M$ be the general linear group, and let $B \subset G$ be the Borel subgroup of upper triangular matrices. The Bruhat decomposition can be written $G = \coprod_{\sigma\in\mathfrak{S}_n} B\sigma B$ where $\mathfrak{S}_n$ is the symmetric group. For $\sigma \in \mathfrak{S}_n$ let $T_\sigma = \frac{1}{|B|}\sum_{x\in B\sigma B} x \in \mathbb{C}G$. The Hecke ring is the $\mathbb{Z}$-ring spanned by the T_σ and denoted by $\mathcal{H}_{\mathbb{C}}(G, B)$. Iwahori then proved that this definition can be extended for q outside of prime powers and this Hecke algebra, denoted by $\mathcal{H}_n(q)$, has a presentation given by generators $T_1, \dots, T_{n-1}$ (where $T_i := T_{s_i}$) and relations

$$\begin{array}{lll} (1) & T_i^2 = q \cdot 1 + (q-1)T_i, & 1 \le i \le n-1, \\ (2) & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & 1 \le i \le n-2, \\ (3) & T_iT_j = T_jT_i & \text{if } |i-j| \ge 2. \end{array}$$

L. Solomon [12] studied the Iwahori algebra $\mathcal{H}_{\mathbb{C}}(M, B)$. The Bruhat decomposition is now $M = \coprod_{r\in R_n} BrB$ where R_n is the set of *rook matrices* consisting of $n \times n$ matrices with entries $\{0, 1\}$ and at most one nonzero entry in each row and column; see [11]. Those matrices form a monoid (a generalization of groups in which the elements do not have to be invertible), called the rook monoid. In [13], Solomon described a generalization $\mathcal{I}_n(q)$ outside of prime powers and a first presentation, which Halverson reformulated in [5] into the following one, with generators $T_1, \dots, T_{n-1}, P_1, \dots, P_n$ together with Relations (1) (2) and (3) above and extra relations:

$$\begin{array}{lll} (4) & P_i^2 = P_i, & 1 \leq i \leq n, \\ (5) & P_i P_j = P_j P_i, & 1 \leq i, j \leq n, \\ (6) & P_i T_j = T_j P_i, & 1 \leq i < j \leq n, \\ (7) & P_i T_j = T_j P_i = q P_i & 1 \leq j < i \leq n, \\ (8) & P_{i+1} = q P_i T_i^{-1} P_i & 1 \leq i < n. \end{array}$$

The 0-Hecke algebra is the degeneracy at $q = 0$ of the Hecke algebra $\mathcal{H}_n(q)$ of the symmetric group (or more generally a Coxeter group [4]). Its importance comes, among other things, from its action by divided difference operators on polynomials leading to the Demazure character formula. Starting from the work of Norton and Carter [9, 2], it has received attention from community ranging from combinatorics, algebraic geometry, representation theory and semi-group theory [8]. The goal of this abstract is to define and study the $q = 0$ degeneracy of $\mathcal{I}_n(q)$.

This abstract is structured as follows: after some background (Section 1), we first define the 0-rook monoid R_n^0 by a presentation (Section 2.1) and describe some left and right faithful actions on so-called rook vectors and polynomials (Section 2.2 and Theorem 3). Using these presentations, we prove that R_n^0 is $\mathcal{J}$-trivial (Theorem 4). In Section 3, we investigate the representation theory of R_n^0 including simple and projective modules. We show (Theorem 7) that R_n^0 is projective over H_n^0 and give an explicit rule for the decomposition numbers (Theorem 8). In Section 4, we discuss some work in progress about the branching graphs and tower of monoids.

1 Background

1.1 Rook Matrices and rook vectors

Definition 1. *A rook matrix is a $n \times n$ matrix with entries $\{0, 1\}$ and at most one nonzero entry in each row and column.*

We encode it by its *rook vector* of size n whose i-th coordinate is 0 if there is no 1 in the i-th column of r, and the index of the row with the 1 in the i-th column otherwise.

Example 1. *The rook matrices* $\begin{pmatrix} 0&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$ *and* $\begin{pmatrix} 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{pmatrix}$ *will be denoted by* 04231 *and* 03041.

Any permutation matrix is a rook matrix, as any permutation is a rook vector. The product of two rook matrices is again a rook matrix, so that they form a finite submonoid of the monoid of matrices. We denote by $\mathbb{I}_n$ the identity rook matrix (or its rook vector).

1.2 Representation theory of $\mathcal{J}$-trivial monoid

In 1951, Green introduced several preorders on monoids (see for example: [10, Chapter V]) related to inclusion of ideals. In the following, we write $\mathcal{R}$ for right ideal, $\mathcal{L}$ for left and $\mathcal{J}$ for bi-sided. Let $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$ and M be a monoid. For $x, y \in M$, we write $x \leq_{\mathcal{K}} y$ when the $\mathcal{K}$-ideal generated by x is contained in the $\mathcal{K}$-ideal generated by y. For example, if $\mathcal{K} = \mathcal{L}$, this means that $x \leq_{\mathcal{L}} y$ if $Mx \subset My$ or equivalently if $x = uy$ for some $u \in M$. These relations are clearly preorders and naturally give rise to equivalence relations. For example $x \,\mathcal{L}\, y$ if and only if $Mx = My$. A monoid M is called $\mathcal{K}$-trivial if all $\mathcal{K}$-classes are of cardinality one, that is if the preorders are actual orders. For finite monoid, $\mathcal{R}, \mathcal{L}$ and $\mathcal{J}$ are related as follows:

Lemma 1 ([10] V. Theorem 1.9)**.** A finite *monoid is $\mathcal{J}$-trivial when it is both $\mathcal{R}$ and $\mathcal{L}$-trivial.*

The representation theory of $\mathcal{J}$-trivial monoids has been well studied by Denton, Hivert, Schilling and Thiéry [3]. It turns out that it is combinatorial: more precisely, one can compute the simple, projective modules, the Cartan matrix and even the quiver by computing only in the monoid, without requiring linear combinations. We only summarize very shortly the result here. We denote by $E(M)$ the set of idempotents (elements such that $e^2 = e$) of M.

Theorem 1 ([3])**.** *Let M be a $\mathcal{J}$-trivial monoid. There are as many as simple modules S_e as idempotents $e \in E(M)$, all of dimension 1. Their structure is as follows: S_e is spanned by some ϵ_e with the action of any $m \in M$ given by $m \cdot \epsilon_e = \epsilon_e$ if $me = e$, and 0 otherwise.*

To describe the projective module, define

$$\operatorname{rfix}(x) := \min\{e \in E(M) \mid ex = x\}, \tag{1.1}$$

the min being taken for the $\mathcal{J}$-order.

Theorem 2 ([3])**.** *For any idempotent e denote by $L(e) := Me$, and we set*

$$L_{=}(e) := \{x \in Me \mid \operatorname{rfix}(x) = e\} \quad \text{and} \quad L_{<}(e) := \{x \in Me \mid \operatorname{rfix}(x) <_{\mathcal{L}} e\}\,. \tag{1.2}$$

Then, the projective module P_e associated to S_e is isomorphic to $\mathbb{K}L(e)/\mathbb{K}L_{<}(e)$. In particular, taking as basis the image of $L_{=}(e)$ in the quotient, the action of $m \in M$ on $x \in L_{=}(e)$ is given by: $m \cdot x = mx$ if $\operatorname{rfix}(mx) = e$ and 0 otherwise.

1.3 The 0-Hecke algebra as the algebra of a $\mathcal{J}$-trivial monoid

By putting $q = 0$ in Relation (1) defining the Hecke algebra, one gets the quadratic equation $T_i^2 = -T_i$, the braid relations being unchanged. Further putting $\pi_i := T_i + 1$, the algebra $H_n(0)$ becomes the algebra of a monoid H_n^0 generated by the $(\pi_i)_{i<n}$ with

the relation $\pi_i^2 = \pi_i$ and the braid relations. It turns out that H_n^0 is $\mathcal{J}$-trivial and that the representation theory of $H_n(0)$ worked out by Norton and Carter [9, 2] can be obtained from the general representation theory of $\mathcal{J}$-trivial monoids [3]. Note that, to get a monoid for $H_n(0)$, a common choice is to put $\pi_i := -T_i$. However, this choice does not extend to the rook case.

The key to the representation theory of H_n^0 is the following:

Lemma 2. *Let $x \in H_n^0$. Then, the idempotent $\mathrm{rfix}(x)$ is the maximal element of the parabolic subgroup generated by the π_i's where i is a right descent of x, that is $x\pi_i = x$.*

We therefore recover the fact that the projective modules of H_n^0 have their bases indexed by permutations with a given descent set.

Note 1. *The reader has to be careful that we are working with the π_i basis whereas in the literature it is customary to work with the T_i basis. As a consequence the eigenvalues 0 and -1 with T_i becomes respectively 1 and 0. The usual simple and projective modules for $H_n(0)$ associated with the set $I \subset [\![1, n-1]\!]$ are associated with $[\![1, n-1]\!] \setminus I$ in our conventions.*

2 Definitions and elementary properties

We now define the 0-rook monoid R_0^n by extending the definition of H_0^n.

2.1 Relations

In the relations defining $\mathcal{I}_n(q)$ let $q = 0$ and let $\pi_i = T_i + 1$. We get the quadratic equations $\pi_i^2 = \pi_i$, the braid relations for π_i where $1 \leq i \leq n-1$ together with

$$\begin{array}{lll}
(4) & P_i^2 = P_i, & 1 \leq i \leq n, \\
(5) & P_iP_j = P_jP_i, & 1 \leq i, j \leq n, \\
(6) & P_i\pi_j = \pi_jP_i, & 1 \leq i < j \leq n, \\
(7) & P_i\pi_j = \pi_jP_i = P_i, & 1 \leq j < i \leq n, \\
(8) & P_{i+1} = P_i\pi_iP_i, & 1 \leq i < n.
\end{array}$$

Let R_n^0 be the monoid generated by the generators $\pi_1, \dots \pi_{n-1}, P_1, \dots P_n$ and these relations. The latter clearly show that it is generated only by $P_1, \pi_1, \dots \pi_{n-1}$, and that the Relation (8) is rather a definition. Furthermore, P_n is the *zero* of R_n^0, that is for any $x \in R_n^0$, one has $xP_n = P_nx = P_n$.

By induction one can show that putting $\pi_0 = P_1$, the following is an alternative presentation of R_n^0:

$$\begin{array}{lll}
(1) & \pi_i^2 = \pi_i, & 0 \leq i \leq n-1, \\
(2) & \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1} & 1 \leq i \leq n-2, \\
(3) & \pi_1\pi_0\pi_1\pi_0 = \pi_0\pi_1\pi_0 = \pi_0\pi_1\pi_0\pi_1 & \\
(4) & \pi_i\pi_j = \pi_j\pi_i & \text{if } |i-j| \geq 2.
\end{array}$$

This shows that R_n^0 is a quotient of the Hecke-monoid at $q = 0$ of type B, and that the Hecke-monoid $H_n(q)$ at $q = 0$ of type A is a submonoid of it.

2.2 Acting on vectors and polynomials

Let $r = r_1 \dots r_n \in R_n$. The classical right action of H_n^0 on vectors can be extended to R_n^0 on rook vectors as follows:

$$(r_1 \dots r_n) \cdot \pi_i = \begin{cases} r_1 \dots r_{i-1} r_{i+1} r_i r_{i+2} \dots r_n & \text{if } r_i < r_{i+1}, \\ r_1 \dots r_n & \text{otherwise,} \end{cases} \quad \text{for } 1 \le i \le n-1.$$
$$(r_1 \dots r_n) \cdot \pi_0 = 0 r_2 \dots r_n.$$

Lemma 3. *The previous definition is a right monoid action of R_n^0 on R_n called the* right natural action*. Under this action, P_j acts by killing the first j entries: $(r_1 \dots r_n) \cdot P_j = 0 \dots 0 r_{j+1} \dots r_n$.*

Similarly, we now describe a left action on rook vectors: let $r = r_1 \dots r_n \in R_n$. For $0 \le j \le n$, we write $j \in r$ if $j \in \{r_1, \dots, r_n\}$. The left action of $\pi_i \in R_n^0$ on r can be described the following way:

- π_0 replaces 1 by 0 in r if $1 \in r$, and fixes r otherwise.
- For $i > 0$, the action of π_i on r is
  - if $i, i+1 \in r$, call k and l their respective positions. Then π_i fixes r if $l < k$, otherwise it exchanges i and $i+1$.
  - if $i \notin r$ and $i+1 \in r$, then π_i replaces $i+1$ by i.
  - if $i+1 \notin r$ then π_i fixes r.

Lemma 4. *The previous definition is a left monoid action of R_n^0 on R_n called the* left natural action*. Under this action, P_j acts by replacing the entries smaller than j by* 0*.*

Example 2. $\pi_0 \cdot 0342 = 0342$, $\pi_1 \cdot 0342 = 0341$, $\pi_2 \cdot 0342 = 0342$, $\pi_3 \cdot 0342 = 0432$, $\pi_0 \cdot 132 = 032$.

This sheds some light on the link with the type B: it is well known that type B can be realized using signed permutations. The quotient giving the 0-rook monoid can be realized by replacing the negative numbers by zeros.

One can also extend the action of H_n^0 by isobaric divided differences on polynomials: the monoid R_n^0 acts also on the polynomials in n indeterminates over any ring k, $k[X_1, \dots, X_n]$ in the following way. Let $f \in k[X_1, \dots, X_n]$. Define

$$f \cdot \pi_0 := f_{|X_1=0} = f(0, X_2, \dots X_n), \qquad \text{and} \qquad f \cdot \pi_i := \frac{X_i f - (X_i f) \cdot s_i}{X_i - X_{i+1}}. \tag{2.1}$$

Again, $f \cdot P_j = f(0, \dots 0, X_j, \dots X_n)$. It is actually possible to get an action of the full generic q-rook algebra by letting $f \cdot T_i := q(f \cdot s_i) + (1-q)(f \cdot (\pi_i - 1))$. Since this action is faithful, this leads to very natural definitions of the rook-monoids and algebras.

2.3 Properties of the monoid

In this subsection, we show that the previous actions are faithful, that is the given presentation is equivalent to the definition by operators. We work with the right action proceeding by induction on n, using the chain of inclusions $R_1^0 \subset R_2^0 \subset \cdots \subset R_n^0$. We start by defining a rook analog of the (inverse) Lehmer code of a permutation.

Definition 2. *Let m be the map from the words on $\mathbb{Z}$ to $\mathbb{N}$ defined recursively as follows: $m(\epsilon) = 0$ where ϵ denotes the empty word. For any word $\underline{w}$ and any letter d,*

$$m(\underline{w}d) := \begin{cases} -d & \text{if } d \leq 0\,; \\ m(\underline{w}) + 1 & \text{if } 0 < d \leq m(\underline{w}) + 1\,; \\ m(\underline{w}) & \text{if } d > m(\underline{w}) + 1\,. \end{cases} \tag{2.2}$$

A code *of size n is a word on $\mathbb{Z}$ defined recursively by: the empty word ϵ is a code, and $\underline{w}d$ is a code if $\underline{w}$ is a code and $-m(\underline{w}) \leq d \leq n$. We denote by C_n the set of codes of size n.*

Example 3. *The codes of size* 1, 2 *and* 3 *are the following:* $\{0, 1\}, \{00, 01, 02, 1\bar{1}, 10, 11, 12\}$ *and*

$$\{000, 001, 002, 003, 01\bar{1}, 010, 011, 012, 013, 020, 021, 022, 023, 1\bar{1}\bar{1}, 1\bar{1}0, 1\bar{1}1, 1\bar{1}2,$$
$$1\bar{1}3, 100, 101, 102, 103, 11\bar{2}, 11\bar{1}, 110, 111, 112, 113, 12\bar{2}, 12\bar{1}, 120, 121, 122, 123\}\,.$$

$m(2836427) = 7$, $m(364\bar{4}294\bar{3}52538) = 6$, $m(021\bar{1}1254) = 4$ where $\bar{i}$ stands for $-i$.
The words of C_9 with prefix $021\bar{1}1254$ are $021\bar{1}12544\bar{4}$, $021\bar{1}12543\bar{3}$, ..., $021\bar{1}12549$.

Notation 1. *For $i, n \in \mathbb{N}$ we put:* $\boxed{\begin{smallmatrix} n \\ \vdots \\ i \end{smallmatrix}} := \begin{cases} 1 & \text{if } i > n, \\ \pi_n \dots \pi_i & \text{if } 0 \leq i \leq n, \\ \pi_n \dots \pi_1 \pi_0 \pi_1 \dots \pi_i & \text{if } i < 0. \end{cases}$

Definition 3. *If $\mathsf{c} = \mathsf{c}_1 \dots \mathsf{c}_n \in C_n$, let* $\pi_{\mathsf{c}} := \boxed{\begin{smallmatrix} 0 \\ \vdots \\ \mathsf{c}_1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 1 \\ \vdots \\ \mathsf{c}_2 \end{smallmatrix}} \cdot \dots \cdot \boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}} \in R_n^0$.

Example 4. *Let $\mathsf{c} = 11\bar{1}20$. Then*

$$\pi_{\mathsf{c}} = \boxed{\begin{smallmatrix} 0 \\ \vdots \\ 1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 1 \\ \vdots \\ 1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 2 \\ \vdots \\ -1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 3 \\ \vdots \\ 2 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} = 1 \cdot \pi_1 \cdot \pi_2 \pi_1 \pi_0 \pi_1 \cdot \pi_3 \pi_2 \cdot \pi_4 \pi_3 \pi_2 \pi_1 \pi_0$$

The key fact is that an element of R_n^0 is uniquely determined by its (left or right) action on the rook identity matrix:

Theorem 3. *For all $n \in \mathbb{N}$, the maps $\mathsf{c} \in C_n \mapsto \pi_{\mathsf{c}} \in R_n^0$ and $r \in R_n^0 \mapsto \mathbb{I}_n \cdot r \in R_n$ and $r \in R_n^0 \mapsto r \cdot \mathbb{I}_n \in R_n$ are bijections so that, $|C_n| = |R_n^0| = |R_n|$. In particular any element of R_n^0 can be expressed in a unique way as π_{c}. Moreover these canonical expressions are reduced.*

The map $r \mapsto r \cdot \mathbb{I}_n$, when extended by linearity, is an isomorphism of R_n^0-modules between the left regular module and natural module.

For instance the expression $P_n = \pi_0\pi_1\pi_0\pi_2\pi_1\pi_0\pi_3\pi_2\pi_1\pi_0\dots\pi_{n-1}\pi_{n-2}\dots\pi_1\pi_0$ is the reduced canonical expression of P_n.

Note that, under the compose bijection $\mathsf{c} \mapsto r := \mathbb{I}_n \cdot \pi_{\mathsf{c}}$, the integer $m(\mathsf{c}) + 1$ is the position of the first zero in r. And that the conditions of Definition 2 amount to saying that if a word is reduced, it never exchanges two zeros when applied to $(123\dots n)$.

2.4 Green relations for the rook monoid

In this subsection, we show that R_n^0 is $\mathcal{J}$-trivial. We generalize the notion of inversions of permutations to rooks $r = r_1 \dots r_n \in R_n$ as $\mathrm{Inv}(r) = \{(i,j) \mid i < j \text{ and } r_i > r_j > 0\}$. We also define its support as $\mathrm{Supp}(r) := \{i \mid r_i \neq 0\}$ and its content as $\mathrm{Cont}(r) := \{r_i \neq 0\}$. We order inversions and supports by inclusion, and contents of the same length by product order that is if $\mathrm{Cont}(r) = \{c_1 < \dots < c_l\}$ and $\mathrm{Cont}(r') = \{d_1 < \dots < d_l\}$ we write $\mathrm{Cont}(r) \leq \mathrm{Cont}(r')$ if for all $i \leq l$ one has $c_i \leq d_i$. We then define a relation $\preccurlyeq$ over R_n. If $r, r' \in R_n$ then

$$r \preccurlyeq r' \iff \begin{cases} \mathrm{Supp}(r) \subsetneq \mathrm{Supp}(r'), \text{ or} \\ \mathrm{Supp}(r) = \mathrm{Supp}(r') \text{ and } \mathrm{Cont}(r) \leq \mathrm{Cont}(r') \text{ and } \mathrm{Inv}(r) \supseteq \mathrm{Inv}(r'). \end{cases}$$

It is easy to see that this is an order on R_n, with $\mathbb{I}_n$ as maximal element and $0_n = 00...0$ as minimal one.

Proposition 1. *The left action is regressive: for $f \in R_n^0$ and $r \in R_n$, one has $f \cdot r \preccurlyeq r$. As a consequence, R_n^0 is $\mathcal{L}$-trivial.*

It is well known (see eg: [3]) that any monoid that has a faithful regressive left action is $\mathcal{L}$-trivial. This shows that the $\mathcal{L}$-preorder on R_n^0 is actually an order. It is a rook analog of the weak order for permutations (also called permutohedron order). It is also a lattice. However, it is not a lattice quotient of the type B weak order.

Furthermore, the presentation of R_n^0 is symmetric: thus this monoid is isomorphic to its opposite and is thus also $\mathcal{R}$-trivial. Finally we have proved the following:

Theorem 4. *The monoid R_n^0 is $\mathcal{J}$-trivial.*

3 Representation theory

3.1 Simple modules

As explained in Section 1.2 the representation theory of R_n^0 is governed by its idempotents, as any $\mathcal{J}$-trivial monoid. The following theorem, obtained by writing explicitly the canonical expression of the elements π_J, describes them:

Theorem 5. *The monoid R_n^0 has 2^n idempotents: the zero (maximal element) of every parabolic submonoid. Let $J \subset \{\pi_0, \ldots, \pi_{n-1}\}$, and π_J the zero of the submonoid generated by J. It is an idempotent, and furthermore $\pi_J \cdot \pi_i = \pi_i \cdot \pi_J = \pi_J$ if and only if $\pi_i \in J$.*

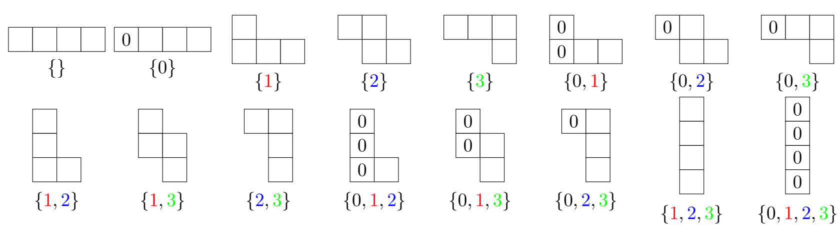
The monoid R_n^0 has 2^n simple modules, all one-dimensional, labeled by the parabolic submonoids. More precisely, $(S_{\pi_J})_{J \subset \{\pi_0, \ldots, \pi_{n-1}\}}$ is a complete set of pairwise non-isomorphic representatives of isomorphism classes of simple R_n^0-modules, where S_{π_J} is defined in Theorem 1.

3.2 Projective modules

Definition 4. *For $\pi \in R_n^0$, we define its left R-descent set (respectively its right R-descent set) by $D_L(\pi) = \{0 \leq i \leq n-1 \mid \pi_i \pi = \pi\}$ (respectively $D_R(\pi) = \{0 \leq i \leq n-1 \mid \pi \pi_i = \pi\}$). A set of all elements of R_n^0 having the same descent sets is called a* R-descent class.

Unless explicitly stated, all R-descents are on the right. There is only a finite number of different R-descent sets possible for $\pi \in R_n^0$: we call them *R-descent type*. We say that a R-descent type is of type 0 if it has 0 in it, 1 otherwise. Similarly to H_n^0, we represent the R-descent type by ribbons with either the first column filled with 0 if it is a 0-type, or empty otherwise. We say that a descent class D is finer than D' if $D \supset D'$.

Example 5. *Here is the list of the R-descent types for R_4^0.*



As an application of Theorem 2 we get :

Theorem 6. *The projective indecomposable R_n^0-modules are indexed by the R-descent type D, and P_D is spanned by the rooks belonging to the descent class D and is isomorphic to the quotient of the associated R-descent class by the finer R-descent classes.*

See the picture at the left of Figure 1 for an example of a projective indecomposable R_n^0-module.

3.3 Restriction to H_n^0

We now investigate the relation between H_n^0 and R_n^0. The restriction of simple modules from R_n^0 to H_n^0 and the induction of simple and projective modules from H_n^0 to R_n^0 are described by simple natural rules, which we don't describe here for space. More interesting are the restriction of projective modules from R_n^0 to H_n^0. Indeed, we show that they are projective on H_n^0 and give a precise combinatorial rule.

Definition 5. *Let $I \subset \{1, \dots, n\}$ of cardinality i. Let $\sigma = i_1 \dots i_n \in \mathfrak{S}_n$. We define $\varphi_I(\sigma)$ to be the rook obtained by removing the first i entries of σ and inserting zeros in positions indexed by the elements of I. We call $\psi : R_n \to \mathfrak{S}_n$ which takes a rook, put all zeros at the beginning of the word and replace them by the missing letters in decreasing order. Then $\psi \circ \varphi_I = id_{\mathfrak{S}_n}$.*

Example 6. *For instance $\varphi_{\{1,3\}}(14235) = 02035$ and $\psi(02410) = 53241$.*

H_n^0 is a submonoid of R_n^0, thus it acts by left multiplication on R_n^0. It therefore makes sense to consider R_n^0 as a H_n^0-module:

Theorem 7. *R_n^0 is projective on H_n^0. As a consequence, any R_n^0-projective module remains projective when restricted to H_n^0.*

Proof. The proof widely uses the isomorphism between the regular R_n^0-module and the natural one (Theorem 3). Since applying H_n^0 does not create any zeros, there is a filtration of R_n^0 in H_n^0-modules depending on the number of zeros of the elements. By projectivity, it is enough to prove that each layer of this filtration is projective. Each layer of this filtration is a direct sum of modules depending on the position of these zeros since the zeros are not moved by the action of H_n^0.

For such a summand where zeros are in positions $i \in I$, the map ψ of Definition 5 is an injective H_n^0-module morphism. Its image is the set of permutations which start with $|I|$ descents which is a well known projective H_n^0-module. □

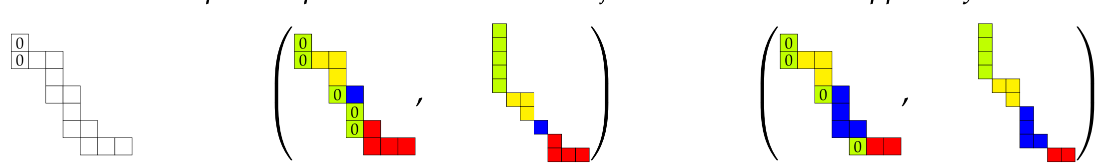
We now describe the restriction functor. We will use the product rule of ribbons (the multiplication rule of ribbon Schur functions in noncommutative symmetric functions [8]). The product of the two ribbons R and S is the sum of the two ribbons obtained by gluing the topmost leftmost box of S either on the right or below the bottommost rightmost box of R.

Definition 6. *Let D be a type of R-descent. Thus D is a set of boxes. A zero-filling of D is a ribbon of shape D with boxes either empty, either with 0 inside according to the following rules:*

- *In the first column, either every box contains 0 if D is of type 0, or none otherwise.*
- *Outside of the first column, if a box contains 0 then there is no box on its left, and all the boxes below also contain zeros.*

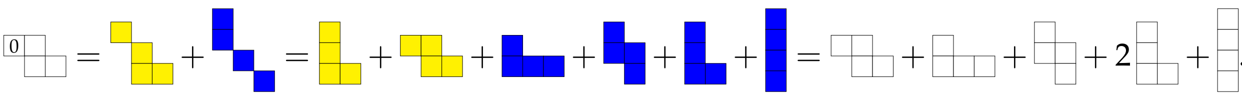
To each of these fillings we associate the product of a column of size the total number of zeros, times the ribbons obtained from D in which each box with a zero on the filling has been removed, in the same order of appearance.

Example 7. *The following picture shows a R-descent type, and two examples of a zero-filling with the associate respective product (the colors are just to show what happens of each box):*



Theorem 8. *The indecomposable projective R_n^0-module of type D splits as a H_n^0-module as the direct sum of all the indecomposable projective H_n^0-module whose descent class are obtained in a product coming from a zero-filling of D, with multiplicity.*

Example 8. *This is an example of decomposition of a indecomposable projective R_n^0-module into indecomposable projective H_n^0-modules. The colors indicate the different products of zero-filling. See Figure 1.*



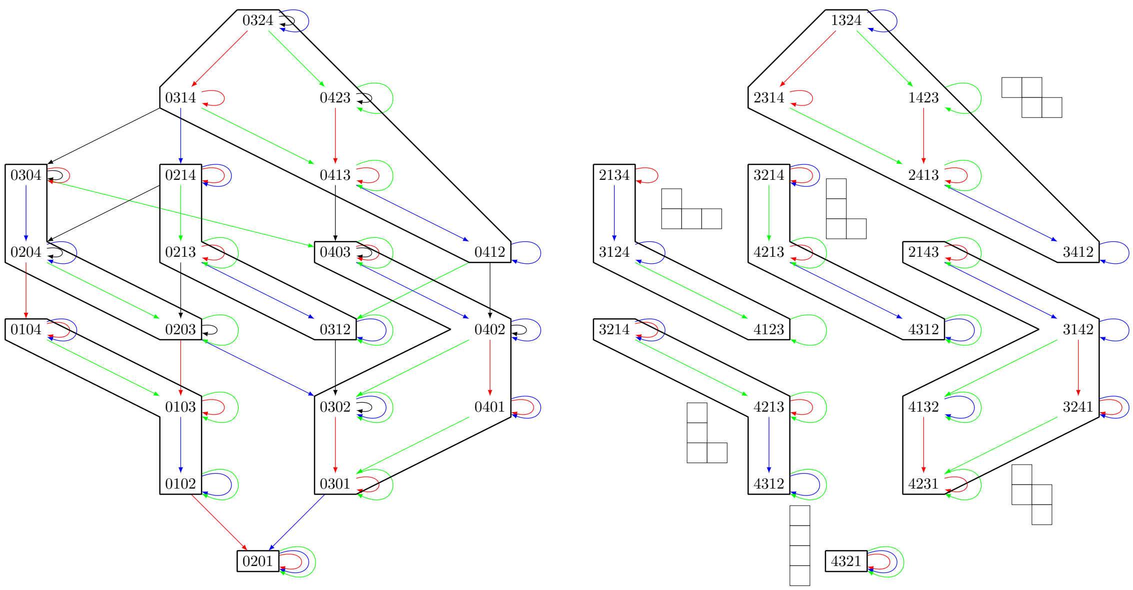
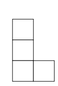
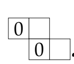


Figure 1: The decomposition of the R_4^0 projective module of type $\{0,2\}$ into H_4^0 projective modules.

We have shown a way to decompose a projective R_n^0-module to projective H_n^0-modules. The following results tells us that we more precisely have a decomposition functor.

Theorem 9. *Let P_R be an indecomposable projective module of R_n^0. Write $P_R = \bigoplus P_H$ its decomposition into indecomposable projective H_n^0-modules. Then the isomorphism of H_n^0-module $\tilde{\varphi} : \bigoplus P_H \to P_R$ is triangular: $\tilde{\varphi}(e) = \varphi_I(e) + \sum_{e'<e} e'$, with φ_I defined in Definition 5 and I the zero-set linked to P_H.*

Example 9. *We know from the Example 8 that there is a*  *module inside the Figure 1, coming from the zero-filling* *. This H_n^0-module is well-known to have the elements* 3214, 4213 *and* 4312. *So ours must contains* $\pi_{\{0,2\}}(3214) = 0104$, $\pi_{\{0,2\}}(4213) = 0103$ *and* $\pi_{\{0,2\}}(4312) = 0102$. *See Figure 1.*

4 Perspectives and towers of monoids

We have the following chain of submonoids: $R_1^0 \subset R_2^0 \subset R_3^0 \subset \cdots \subset R_n^0 \subset \ldots$ We can try to analyse the properties of this chain in term of tower of monoids [1].

The first thing we need to do so is a way to embed $R_n^0 \times R_m^0$ into R_{n+m}^0. We looked at the following embedding which seems more natural regarding the action on rook vectors and polynomials:

$$\begin{array}{ccccc} R_n^0 & \times & R_m^0 & \longrightarrow & R_{n+m}^0 \\ P_i, \pi_j & & & \longmapsto & P_i, \pi_j \\ & & P_i, \pi_j & \longmapsto & P_{n+i}, \pi_{n+j} \,. \end{array}$$

These embeddings are compatible with the usual ones used for H_n^0. Note that they are not injective. Nevertheless they seem to have some nice properties. For example we give here the rule for restriction of simple modules:

Theorem 10. *Let J be a subset of $0, \ldots, n+m$ and S_J the simple R_{n+m}^0-module associated to this parabolic subgroup. Then there are two possibilities for the restriction:*

$$\operatorname{Res}^{R_{n+m}^0}_{R_n^0 \times R_m^0} S_J = \begin{cases} S_{J \cap [0,\ldots,n-1]} \otimes S_{\{0\} \cup (J \cap [n+1,\ldots n-1]) - n} & \text{if } [0,\ldots,n] \subset J, \\ S_{J \cap [0,\ldots,n-1]} \otimes S_{(J \cap [n+1,\ldots n-1]) - n} & \text{otherwise.} \end{cases}$$

It is also possible to compute the induction using Virmaux result [14].

For projective modules, the situation is not so nice: In general, R_n^0 is not projective over R_{n-1}^0 and R_{n+m}^0 is not projective over $R_n^0 \otimes R_m^0$. However, $R_{n+m}^0$is projective over $R_n^0 \otimes H_m^0$. This gives us a structure of noncommutative symmetric function module and co-module on the sum of the Grothendieck group of the category of projective modules $\bigoplus_n K_0(R_n)$, similar to the case of $H_n^0(B)$. [6]

It is worth noticing that the embedding we choose is not the only one. There are other choices of what happens to π_0 of the right-side. We choose to send it to P_{n+1}, but sending it to 1, π_0 or to $0 = P_{n+m}$ also gives other embeddings. All of these embeddings are non-injective, but it seems that we could also use some combinations of these choices. We are currently searching for an embedding for which the projectivity property holds.

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